Exercise 2.10 (Solutions)

Question # 1

Let x and $30-x$ be two positive integers and P denotes product integers then

$$P = x(30-x)$$
$$= 30x - x^2$$

Diff. w.r.t. x

$$\frac{dP}{dx} = 30-2x \;\ldots\ldots\; (i)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = -2 \;\ldots\ldots\; (ii)$$

For critical points, put $\frac{dP}{dx}=0$

$$\Rightarrow\; 30-2x=0 \quad \Rightarrow\; -2x=-30 \quad \Rightarrow\; x=15$$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=2} = -2 < 0$$

$\Rightarrow$ P is maximum at $x=15$

Other $+$tive integer $= 30-x = 30-15 = 15$

Hence 15 and 15 are the required positive numbers.

Question # 2

Let x be the part of 20 then other is $20-x$.

Let S denotes sum of squares then

$$S = x^2 + (20-x)^2$$
$$= x^2 + 400 - 40x + x^2$$
$$= 2x^2 - 40x + 400$$

Diff. w.r.t x

$$\frac{dS}{dx} = 4x-40 \;\ldots\ldots\; (i)$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 4 \;\ldots\ldots\; (ii)$$

For stationary points put $\frac{dS}{dx}=0$

$$\Rightarrow\; 4x-40=0 \quad \Rightarrow\; 4x=40 \quad \Rightarrow\; x=10$$

Putting value of x in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=10} = 4 > 0$$

$\Rightarrow$ S is minimum at $x=10$

Other integer $= 20-x = 20-10 = 10$

Hence 10, 10 are the two parts of 20.

Question # 3

Let x and $12-x$ be two $+$tive integers and P denotes product of one with square of the other then

$$P = x(12-x)^2$$
$$\Rightarrow\; P = x\left(144-24x+x^2\right)$$
$$= x^3 - 24x^2 + 144x$$

Diff. w.r.t x

$$\frac{dP}{dx} = 3x^2 - 48x + 144 \;\ldots\ldots\; (i)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 6x-48 \ \ldots\ldots. (ii)$$

For critical points put $\frac{dP}{dx}=0$

$$3x^2-48x+144 = 0$$

$\Rightarrow x^2-16x+48 = 0 \quad \Rightarrow x^2-4x-12x+48 = 0$

$\Rightarrow x(x-4)-12(x-4) = 0 \quad \Rightarrow (x-4)(x-12) = 0$

$\Rightarrow x=4$ or $x=12$

We can not take $x=12$ as sum of integers is 12. So put $x=4$ in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=4} = 6(4)-48$$

$$= 24-48 = -24 < 0$$

$\Rightarrow P$ is maximum at $x=4$.

So the other integer $= 12-4 = 8$

Hence 4, 8 are the required integers.

Question # 4

Let the remaining sides of the triangles are x and y

Perimeter $= 16$

$\Rightarrow 6+x+y = 16$

$\Rightarrow x+y = 16-10 \quad \Rightarrow x+y = 6 \quad \Rightarrow y=10-x \ \ldots\ldots.. (i)$

Now suppose A denotes the square of the area of triangle then

$$A = s(s-a)(s-b)(s-c)$$

Where $s=\frac{a+b+c}{2}=\frac{6+x+y}{2}$

$=\frac{6+x+10-x}{2}$ from (i)

$=\frac{16}{2} = 8$

So $A = 8(8-6)(8-x)(8-y)$

$= 8(2)(8-x)(8-10+x) = 16(8-x)(-2+x)$

$= 16(-16+2x+8x-x^2)$

$\Rightarrow A = 16(-16+10x-x^2)$

Diff. w.r.t x

$$\frac{dA}{dx} = 16(10-2x) \ \ldots\ldots\ldots (i)$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = 16(-2) = -32$$

For critical points put $\frac{dA}{dx}=0$

$16(10-2x) = 0 \Rightarrow (10-2x) = 0 \Rightarrow -2x = -10 \Rightarrow x=5$

Putting value of x in (ii)

$$\left.\frac{d^2A}{dx^2}\right|_{x=5} = -32 < 0$$

$\Rightarrow A$ is maximum at $x=5$

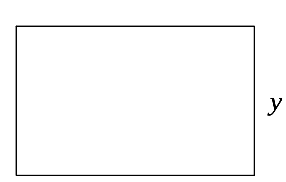
Putting value of x in (i)

$y = 10-5 = 5$

Hence length of remaining sides of triangles are $5cm$ and $5cm$.

Question # 5

Let x and y be the length and breadth of rectangle, then



Area $= A = xy \ \ldots\ldots.. (i)$

Perimeter $= 60$

$\Rightarrow x+x+y+y = 60 \quad \Rightarrow 2x+2y = 60$

$\Rightarrow\ x+y = 30 \quad \Rightarrow\ y = 30-x$ ……. (ii)

Putting in (i)

$$A = x(30-x) \quad \Rightarrow \quad A = 30x-x^2$$

Diff. w.r.t x

$$\frac{dA}{dx} = 30-2x \text{ ……….. } (iii)$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = -2 \text{ ………… } (iv)$$

For critical points put $\frac{dA}{dx}=0$

$$30-2x = 0 \quad \Rightarrow \quad -2x = -30 \quad \Rightarrow \quad x = 15$$

Putting value of x in (iv)

$$\left.\frac{d^2A}{dx^2}\right|_{x=15} = -2 < 0$$

$\Rightarrow$ A is maximum at $x=15$

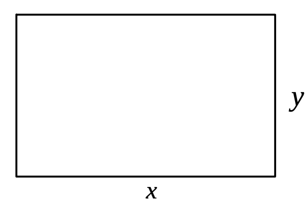
Putting value of x in (ii)

$$y = 30-15 = 15$$

Hence dimension of rectangle is $15cm$, $15cm$.

Question # 6

Let x and y be the length and breadth of the rectangle then



Area $= xy$

$$\Rightarrow\ 36 = xy \quad \Rightarrow \quad y = \frac{36}{x} \text{ ………… } (i)$$

Now perimeter $= 2x+2y$

$$\Rightarrow\ P = 2x+2\left(\frac{36}{x}\right)$$

$$= 2\left(x+36x^{-1}\right)$$

Diff. P w.r.t x

$$\frac{dP}{dx} = 2\left(1-36x^{-2}\right) \text{ ………. } (ii)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 2\left(0-36\left(-2x^{-3}\right)\right) = 2\left(72x^{-3}\right) = \frac{144}{x^3}$$

For critical points put $\frac{dP}{dx}=0$

$$2\left(1-36x^{-2}\right)=0 \quad \Rightarrow \quad 1-\frac{36}{x^2}=0 \quad \Rightarrow \quad 1=\frac{36}{x^2} \quad \Rightarrow \quad x^2 = 36 \quad \Rightarrow \quad x = \pm 6$$

Since length can not be negative therefore $x=6$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=6} = \frac{144}{(6)^3} > 0$$

Hence P is minimum at $x=6$.

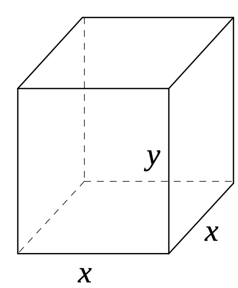
Putting in eq. (i)

$$y = \frac{36}{6} = 6$$

Hence $6cm$ and $6cm$ are the lengths of the sides of the rectangle.

Question # 7

Let x be the lengths of the sides of the base and y be the height of the box.



Then Volume $= x\cdot x\cdot y$

$$\Rightarrow\ 4 = x^2y \quad \Rightarrow \quad y = \frac{4}{x^2} \text{ ……… } (i)$$

Suppose S denotes the surface area of the box, then

$$S = x^2+4xy$$

$\Rightarrow S = x^2 + 4x\left(\dfrac{4}{x^2}\right) \quad \Rightarrow S = x^2 + 16x^{-1}$

Diff. S w.r.t x

$\dfrac{dS}{dx} = 2x - 16x^{-2}$ ……….. (ii)

Again diff. w.r.t x

$\dfrac{d^2S}{dx^2} = 2 - 16\left(-2x^{-3}\right) = 2 + \dfrac{32}{x^3}$ ………… (iii)

For critical points, put $\dfrac{dS}{dx} = 0$

$2x - 16x^{-2} = 0 \quad \Rightarrow 2x - \dfrac{16}{x^2} = 0 \quad \Rightarrow \dfrac{2x^3 - 16}{x^2} = 0$

$\Rightarrow 2x^3 - 16 = 0 \quad \Rightarrow 2x^3 = 16 \quad \Rightarrow x^3 = 8 \quad \Rightarrow x = 2$

Putting in (ii)

$\left.\dfrac{d^2S}{dx^2}\right|_{x=2} = 2 + \dfrac{32}{(2)^3} > 0$

$\Rightarrow S$ is min. when $x = 2$

Putting value of x in (i)

$y = \dfrac{4}{(2)^2} = 1$

Hence 2*dm*, 2*dm* and 1*dm* is the dimension of the box.

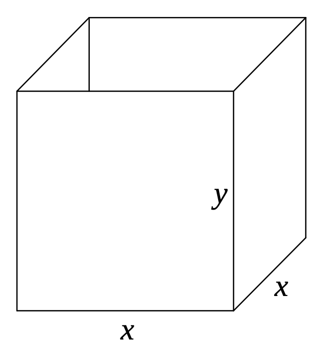
Question # 8 *Do yourself as question # 5.*

Question # 9

Let y be the height of the open tank.

Then Volume = $x \cdot x \cdot y$

$\Rightarrow V = x^2 y \quad \Rightarrow y = \dfrac{V}{x^2}$ ……….. (i)



If S denotes the surface area the open tank, then

$S = x^2 + 4xy$

$= x^2 + 4x\left(\dfrac{V}{x^2}\right) \quad \Rightarrow S = x^2 + 4Vx^{-1}$

Diff. w.r.t x

$\dfrac{dS}{dx} = 2x - 4Vx^{-2}$ …….. (ii)

Again diff. w.r.t x

$\dfrac{d^2S}{dx^2} = 2 - 4V\left(-2x^{-3}\right) = 2 + \dfrac{8V}{x^3}$ ………… (iii)

For critical points, put $\dfrac{dS}{dx} = 0$

$2x - 4Vx^{-2} = 0 \quad \Rightarrow 2x - \dfrac{4V}{x^2} = 0 \quad \Rightarrow \dfrac{2x^3 - 4V}{x^2} = 0 \quad \Rightarrow 2x^3 - 4V = 0$

$\Rightarrow 2x^3 = 4V \quad \Rightarrow x^3 = 2V \quad \Rightarrow x = (2V)^{\frac{1}{3}}$

Putting in (ii)

$\left.\dfrac{d^2S}{dx^2}\right|_{x=(2V)^{\frac{1}{3}}} = 2 + \dfrac{8V}{\left((2V)^{\frac{1}{3}}\right)^3} = 2 + \dfrac{8V}{2V} = 2 + 4 = 6 > 0$

$\Rightarrow S$ is minimum when $x = (2V)^{\frac{1}{3}}$ i.e. $x^3 = 2V \quad \Rightarrow V = \dfrac{x^3}{2}$

Putting in (i)

$y = \dfrac{x^3/2}{x^2} = \dfrac{x}{2}$

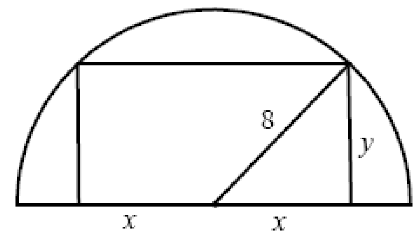
Hence height of the open tank is $\dfrac{x}{2}$.

Question # 10

Let $2x$ & y be dimension of rectangle.

Then from figure, using Pythagoras theorem

$$x^2 + y^2 = 8^2 \quad \Rightarrow \quad y^2 = 64 - x^2 \quad \ldots\ldots\ldots (i)$$



Now Area of the rectangle is given by

$$A = 2x \cdot y$$

Squaring both sides

$$A^2 = 4x^2y^2$$
$$= 4x^2\left(64 - x^2\right)$$
$$= 256x^2 - 4x^4$$

Now suppose $f = A^2 = 256x^2 - 4x^4$ ……….. (ii)

Diff. w.r.t x

$$\frac{df}{dx} = 512x - 16x^3 \quad \ldots\ldots\ldots (iii)$$

Again diff. w.r.t x

$$\frac{d^2f}{dx^2} = 512 - 48x^2 \quad \ldots\ldots\ldots (iv)$$

For critical points, put $\frac{df}{dx} = 0$

$$\Rightarrow 512x - 16x^3 = 0$$
$$\Rightarrow 16x\left(32 - x^2\right) = 0$$
$$\Rightarrow 16x = 0 \quad \text{or} \quad 32 - x^2 = 0$$
$$\Rightarrow x = 0 \quad \text{or} \quad x^2 = 32$$
$$\Rightarrow x = \pm 4\sqrt{2}$$

Since x can not be zero or $-$ive, therefore

$$x = 4\sqrt{2}$$

Putting in (iv)

$$\left.\frac{d^2f}{dx^2}\right|_{x=4\sqrt{2}} = 512 - 48\left(4\sqrt{2}\right)^2$$
$$= 512 - 48(32) = 512 - 1536 = -1024 < 0$$

$\Rightarrow$ Area is max. for $x = 4\sqrt{2}$

Hence length $= 2x = 2\left(4\sqrt{2}\right)$

Breadth $= y = \sqrt{64 - \left(4\sqrt{2}\right)^2} = \sqrt{64 - 32} = \sqrt{32} = 4\sqrt{2}$

Hence dimension is $8\sqrt{2}$ cm and $4\sqrt{2}$ cm.

Question # 11

Let $P(x, y)$ be point and let $A(3, -1)$

Then $d = |AP| = \sqrt{(x-3)^2 + (y+1)^2}$

$$\Rightarrow d^2 = (x-3)^2 + (y+1)^2$$
$$= (x-3)^2 + \left(x^2 - 1 + 1\right)^2 \qquad \because\ y = x^2 - 1 \text{ (given)}$$
$$\Rightarrow d^2 = (x-3)^2 + x^4$$

Let $f = d^2 = (x-3)^2 + x^4$

Diff. w.r.t x

$$\frac{df}{dx} = 2(x-3)+4x^3 \quad \ldots\ldots\ldots\ldots (i)$$

Again diff. w.r.t x

$$\frac{d^2 f}{dx^2} = 2+12x^2 \quad \ldots\ldots\ldots (ii)$$

For stationary points, put $\frac{df}{dx} = 0$

$$2(x-3)+4x^3 = 0$$

$$\Rightarrow \; 2x-6+4x^3 = 0$$

$$\Rightarrow \; 4x^3+2x-6 = 0$$

$$\Rightarrow \; 2x^3+x-3 = 0 \qquad \div \text{ing by } 2$$

By synthetic division

1	2	0	1	-3
	↓	2	2	3
	2	2	3	0

$$\Rightarrow \; x = 1 \quad \text{or} \quad 2x^2+2x+3 = 0$$

$$\Rightarrow \; x = \frac{-2 \pm \sqrt{4-4(2)(3)}}{4}$$

$$= \frac{-2 \pm \sqrt{-20}}{4}$$

This is complex and not acceptable.

Now put $x=1$ in (ii)

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = 2+12(1)^2 = 14 > 0$$

$\Rightarrow \; d$ is maximum at $x=1$.

$$y = 1^2 - 1 = 0$$

$\because \; (1,0)$ is the required point.

Question # 12

Do yourself as Q # 11

THE END

اس تصویر کے درمیان چار نقطوں کو ۱۵ سیکنڈ کے لئے غور سے دیکھیے۔ پھر اپنی کسی قریبی دیوار کی طرف دیکھتے ہوئے تیزی سے آنکھیں جھپکیں۔

آپ کو ایک بزرگ کی تصویر نظر آئے گی۔۔۔۔۔ پہچانیے تو یہ کون ہیں۔